An Area Law for the Entanglement entropy of Eigen-States in a Disordered XY-chain

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$$H_n = -\sum_{j=1}^{n-1} \mu_j[(1+\gamma_j)\sigma_j^X \sigma_{j+1}^X + (1-\gamma_j)\sigma_j^Y \sigma_{j+1}^Y] - \sum_{j=1}^{n} \nu_j \sigma_j^Z$$

- Hilbert space $\mathcal{H} = \bigotimes_{j\in\Lambda} \mathbb{C}^2$ corresponds to the spin system on $\Lambda = [1,n] \cap \mathbb{Z}$.
- $\gamma_j \in [0,1]$, $\mu_j > 0$.
- ν_j's are i.i.d. random variable from a “nice” distribution.

- Define:

$$a_j := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j, \text{ and } a_j^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_j. \tag{1}$$

  Note that

$$a^* a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad a a^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

- The Jordan-Wigner operators

$$c_j := \sigma_1^Z \ldots \sigma_{j-1}^Z a_j, \text{ and } c_j^* = \sigma_1^Z \ldots \sigma_{j-1}^Z a_j^*, \quad j = 1, \ldots, n, \tag{3}$$

- The c_j and c_j^* satisfy the *canonical anti-commutation relations* (CAR),

$$\{c_j, c_k^*\} = \delta_{jk}\, \mathbb{1}, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0 \quad \text{for all } j, k = 1, \ldots, n, \tag{4}$$

$$H_n = \mathcal{C}^* M \mathcal{C}, \tag{5}$$

where

$$M_n = \begin{pmatrix} A_n & B_n \\ -B_n & -A_n \end{pmatrix}, \tag{6}$$

$$A_n = \begin{pmatrix} -\nu_1 & \mu_1 & & \\ \mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \mu_{n-1} & -\nu_n \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & \gamma_1\mu_1 & & \\ -\gamma_1\mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & \gamma_{n-1}\mu_{n-1} \\ & & -\gamma_{n-1}\mu_{n-1} & 0 \end{pmatrix}.$$

$$\mathcal{C} := (c_1, c_2, \ldots, c_1^*, \ldots, c_n^*)^t$$

- Note that $A^* = A^t = A$ and $B^* = B^t = -B$, and thus $M^* = M^t = M$.
- Let $S = A + B$, then using the SVD $\exists\, U, V$ orthogonal and $\Lambda = \mathrm{diag}\{\lambda_j, j = 1, 2, \ldots, n\}$, such that

$$USV^t = \Lambda \tag{7}$$

Note that

$$W := \frac{1}{2}\begin{pmatrix} V+U & V-U \\ V-U & V+U \end{pmatrix},$$

is a Bogoliubov matrix and it diagonalises M

$$WMW^t = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix}$$

Lemma

$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$ *almost surely.*

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- Recall that $H_n = \mathcal{C}^* M \mathcal{C} = \mathcal{C}^* W^t \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} W \mathcal{C}$, and W is Bogoliubov.
- Define the Fermionic system $\mathcal{B} := W\mathcal{C}$.
- H_n is a free Fermion System in terms of $\mathcal{B}$:

$$H_n = 2\sum_{j=1}^{n} \lambda_j b_j^* b_j - \sum_{j=1}^{n} \lambda_j \, \mathbb{1}, \tag{8}$$

- Let Ω be the unique vacuum state of the b's, then consider the ONB of $\mathcal{H}$

$$\psi_\alpha = (b_1^*)^{\alpha_1} \dots (b_n^*)^{\alpha_n} \Omega, \quad \alpha \in \{0,1\}^n. \tag{9}$$

- $\{\psi_\alpha\}_\alpha$ are the Eigen-states of H_n and the spectrum is

$$\sigma(H_n) = \left\{ \sum_{j:\alpha_j=1} \lambda_j - \sum_{j:\alpha_j=0} \lambda_j \, : \, \alpha \in \{0,1\}^n \right\}.$$

- We consider the decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2$, where

  $$\mathcal{H}_1 = \bigotimes_{j \in \Lambda_0} \mathbb{C}^2,$$

  corresponds to the connected subsystem $\Lambda_0 := \{r, r+1, \ldots, r+\ell-1\} \subset \Lambda$.
- The von Neumann Entanglement Entropy of the eigen-states $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ is defined as follows:

  $$\mathcal{E}(\rho_\alpha) = -\operatorname{Tr} \rho_{\alpha,\ell} \log \rho_{\alpha,\ell}, \quad \text{where} \quad \rho_{\alpha,\ell} = \operatorname{Tr}_{\mathcal{H}_2} \rho_\alpha.$$

Correlation Matrix

The correlation matrix of the Fermionic system $\mathcal{B}$ with respect to a state ρ is defined to be the $2n \times 2n$ matrix

$$\Gamma_\rho^{\mathcal{B}} := \langle \mathcal{B}\mathcal{B}^* \rangle_\rho, \qquad \longrightarrow = \left(\operatorname{Tr} b_j^{\#} b_k^{\#} \rho\right)_{j,k=1,2,\ldots,n,\#\in\{\emptyset,*\}}$$

where $\mathcal{B}$ the Fermionic system on $\mathcal{H}$ that corresponds to the b_j's:

$$\mathcal{B} = \begin{pmatrix} b_1 & b_2 & \ldots & b_n & b_1^* & \ldots & b_n^* \end{pmatrix}^t.$$

Lemma

The correlation matrix of the eigen-state ρ_α *where* $\alpha \in \{0,1\}^n$ *with respect to the Jordan-Wigner Fermionic system* $\mathcal{C}$ *is almost surely the orthogonal projection on*

$$\Delta_\alpha := \{\lambda_j : \alpha_j = 0\} \cup \{-\lambda_j : \alpha_j = 1\},$$

where $\{\lambda_j, -\lambda_j, j = 1, 2, \ldots, n\}$ *are the eigenvalues of the effective one-particle Hamiltonian* M*; i.e.*

$$\Gamma_{\rho_\alpha}^{\mathcal{C}} = \chi_{\Delta_\alpha}(M), \text{ almost surely.}$$

Wick's rule

Let $f, g : [1, n] \to \mathbb{C}$ and $\{b_j\}_{j=1}^{n}$ be a set of Fermionic operators, then define

$$B(f,g) := \sum_{j=1}^{n} \overline{f}_j b_j + \sum_{k=1}^{n} g_k b_k^*, \tag{10}$$

where $f := (f_1, f_2, \ldots, f_n)^t,\ f_j := f(j)$ and $g := (g_1, g_2, \ldots, g_n)^t,\ g_j := g(j)$ Let ρ be self adjoint on $\mathcal{B}(\mathcal{H})$ with $\operatorname{Tr} \rho = 1$, we say that ρ satisfies Wick's Rule with respect to the set of Fermionic system $\mathcal{B}$ in $\mathcal{H}$ if for any positive integer m

$$\left\langle \prod_{j=1}^{m} B_j \right\rangle_\rho = \begin{cases} 0, & \text{if } m \text{ is odd;} \\ \sum_{k=2}^{m} (-1)^k \langle B_k B_1 \rangle_\rho \left\langle \prod_{\substack{j=2 \\ j \neq k}}^{m} B_j \right\rangle_\rho, & \text{if } m \text{ is even.} \end{cases} \tag{11}$$

Recall that

$$\mathcal{E}(\rho_\alpha) = -\operatorname{Tr}\rho_{\alpha,\ell}\log\rho_{\alpha,\ell}, \quad \text{where} \quad \rho_{\alpha,\ell} = \operatorname{Tr}_{\mathcal{H}_2}\rho_\alpha.$$

Lemma

Let $\mathcal{B}$ be a Fermionic system in $\mathcal{H}$, ρ is any self adjoint operator in $\mathcal{B}(\mathcal{H})$ with $\operatorname{Tr}\rho = 1$. If ρ satisfies Wick's rule with respect to $\mathcal{B}$ then

$$\operatorname{Tr}\rho\log\rho = \operatorname{tr}\Gamma_\rho^{\mathcal{B}}\log\Gamma_\rho^{\mathcal{B}}$$

- Define the local Jordan-Wigner operators $\{c_j^{(1)},\ j\in\Lambda_0\}$ on $\mathcal{H}_1$ as follows

$$c_r^{(1)}=a\otimes\mathbb{1}^{\otimes(\ell-1)},\ \ c_j^{(1)}:=\left(\sigma^Z\right)^{\otimes(j-1)}\otimes a\otimes\mathbb{1}^{\otimes(\ell-j)},\ \ j\in\Lambda_0.$$

- $c_j^{(1)}$'s are Fermionic operators on $\mathcal{H}_1$.

Lemma

Let $\mathcal{C}_\ell$ corresponds to $\{c_j^{(1)},\ j\in\Lambda_0\}$, and P_{Λ_0} be the orthogonal projection on $\operatorname{span}\{e_j,e_{n+j},j\in\Lambda_0\}$, then

$$\Gamma^{\mathcal{C}_\ell}_{\rho_{\alpha,\ell}}=P_{\Lambda_0}\Gamma^{\mathcal{C}}_{\rho_\alpha}P_{\Lambda_0}.$$

Theorem

$\rho_{\alpha,\ell}$ satisfy Wick's rule with respect to the system $\mathcal{C}_\ell$.

Theorem

The von Neumann Entanglement Entropies of the eigen-states $\{\rho_\alpha, \alpha \in \{0,1\}^n\}$ *with respect to the connected subsystem* $\Lambda_0 \subset \Lambda$ *of length* ℓ *are given by the formulas:*

$$\mathcal{E}(\rho_\alpha) = -\operatorname{tr} \Gamma_{\rho_{\alpha,\ell}}^{\mathcal{C}_\ell} \log \Gamma_{\rho_{\alpha,\ell}}^{\mathcal{C}_\ell} \tag{12}$$

Apply the reorder of basis $(e_1, e_2, \ldots, e_n, e_{n+1}, \ldots, e_{2n}) \longrightarrow P := (e_1, e_{n+1}, e_2, e_{2+n}, \ldots, e_n, e_{2n})$, it turns M_n into a unitarely equivalent block Jacobi matrix:

$$\widetilde{M} := \begin{pmatrix} \nu_1 \sigma^Z & -\mu_1 S(\gamma_1) & & \\ -\mu_1 S(\gamma_1)^t & \ddots & \ddots & \\ & \ddots & \ddots & -\mu_{n-1} S(\gamma_{n-1}) \\ & & -\mu_{n-1} S(\gamma_{n-1}) & \nu_n \sigma^Z \end{pmatrix},$$

where

$$S(\gamma) = \sigma^Z + i\sigma^Y = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}$$

Theorem

Suppose that M_n *is bounded uniformly in* n, *and dynamically localized in the sense that there exists* $C < \infty$ *and* $\eta > 0$ *such that*

$$\mathbb{E}\left(\sup_{|g|\leq 1}\left\|\left[g(\widetilde{M_n})\right]_{\mathbf{jk}}\right\|\right) \leq Ce^{-\eta|\mathbf{j}-\mathbf{k}|},$$

for all $n \in \mathbb{N}$ *and* $1 \leq \mathbf{j}, \mathbf{k} \leq n$. *then the Entanglement Entropy of the eigen-states satisfy an area law, i.e. there exists* $0 < C' < \infty$ *such that*

$$\mathbb{E}\left(\sup_{\alpha}\mathcal{E}(\rho_\alpha)\right) < C'.$$

$$-x\log x-(1-x)\log(1-x)\leq 2\log 2\sqrt{x(1-x)},\quad \text{for}\quad 0<x<1.$$

$$\begin{aligned}
\mathcal{E}(\rho_\alpha) &\leq 2\log 2\sum_{j=1}^{\ell}\sqrt{\xi_j^{(\alpha)}(1-\xi_j^{(\alpha)})}\\
&= \log 2\,\mathrm{tr}\left(\Gamma^\ell(\mathbb{1}-\Gamma^\ell)\right)^{\frac{1}{2}}=\log 2\,\mathrm{tr}\left(\widetilde{\Gamma}^\ell(\mathbb{1}-\widetilde{\Gamma}^\ell)\right)^{\frac{1}{2}}\\
&\leq 2\log 2\sum_{j=1}^{2\ell}\left(\left[\widetilde{\Gamma}^\ell(\mathbb{1}-\widetilde{\Gamma}^\ell)\right]_{jj}\right)^{\frac{1}{2}}.
\end{aligned}$$

$$\mathcal{E}(\rho_\alpha)\leq\log 2\sum_{\mathbf{j}\in\boldsymbol{\Lambda}_{\mathbf{0}}}\sqrt{\mathrm{tr}\left[\widetilde{\Gamma}(\mathbb{1}-\widetilde{\Gamma})\right]_{\mathbf{jj}}}$$

Using $\widetilde{\Gamma}^2 = \widetilde{\Gamma}$ with block matrix multiplication to get,

$$\widetilde{\Gamma}_{\mathbf{jj}} = \widetilde{\Gamma}^2_{\mathbf{jj}} + \sum_{\substack{\mathbf{k} \in \mathbf{\Lambda_0} \\ \mathbf{j} \neq \mathbf{k}}} \widetilde{\Gamma}_{\mathbf{jk}} \widetilde{\Gamma}_{\mathbf{kj}} + \sum_{\mathbf{k} \in \mathbf{\Lambda} \setminus \mathbf{\Lambda_0}} \widetilde{\Gamma}_{\mathbf{jk}} \widetilde{\Gamma}_{\mathbf{kj}}.$$

Then for $\mathbf{j} \in \mathbf{\Lambda_0}$, we have,

$$\left[\widetilde{\Gamma}(\mathbb{1} - \widetilde{\Gamma})\right]_{\mathbf{jj}} = \widetilde{\Gamma}_{\mathbf{jj}}(\mathbb{1} - \widetilde{\Gamma}_{\mathbf{jj}}) - \sum_{\substack{\mathbf{k} \in \mathbf{\Lambda_0} \\ \mathbf{k} \neq \mathbf{j}}} \widetilde{\Gamma}_{\mathbf{jk}} \widetilde{\Gamma}_{\mathbf{kj}} = \sum_{\mathbf{k} \in \mathbf{\Lambda_0}} \widetilde{\Gamma}_{\mathbf{jk}} \widetilde{\Gamma}_{\mathbf{kj}}, \quad (13)$$

$$\begin{aligned} \mathcal{E}(\rho_\alpha) &\leq 2 \log 2 \sum_{\mathbf{j} \in \Lambda_0} \left(\sum_{\mathbf{k} \in \Lambda \setminus \Lambda_0} \operatorname{tr} \widetilde{\Gamma}_{\mathbf{jk}} (\widetilde{\Gamma}_{\mathbf{jk}})^t \right)^{\frac{1}{2}} \\ &\leq 2 \log 2 \sum_{\mathbf{j} \in \Lambda_0} \sum_{\mathbf{k} \in \Lambda \setminus \Lambda_0} \left(\operatorname{tr} \widetilde{\Gamma}_{\mathbf{jk}} (\widetilde{\Gamma}_{\mathbf{jk}})^t \right)^{\frac{1}{2}} \end{aligned}$$

Then note that

$$\operatorname{tr}\widetilde{\Gamma}_{\mathbf{jk}}(\widetilde{\Gamma}_{\mathbf{jk}})^t \leq 2\|\widetilde{\Gamma}_{\mathbf{jk}}\|^2,$$

and we get

$$\mathcal{E}(\rho_\alpha) \leq 2\sqrt{2}\log 2 \sum_{\mathbf{j}\in\Lambda_0}\sum_{\mathbf{k}\in\Lambda\setminus\Lambda_0} \|\widetilde{\Gamma}_{\mathbf{jk}}\|$$

Then by taking the $\sup$ over α then averaging, we get

$$\mathbb{E}\left(\sup_\alpha \mathcal{E}(\rho_\alpha)\right) \leq 2\sqrt{2}\log 2 \sum_{\mathbf{j}\in\boldsymbol{\Lambda}_\mathbf{0}}\sum_{\mathbf{k}\in\boldsymbol{\Lambda}\setminus\boldsymbol{\Lambda}_\mathbf{0}} \mathbb{E}\left(\sup_\alpha \left\|\left[\chi_{\Delta_\alpha}(\widetilde{M})\right]_{\mathbf{jk}}\right\|\right)$$

Now,

$$\mathbb{E}\left(\sup_\alpha \left\|\left[\chi_{\Delta_\alpha}(\widetilde{M})\right]_{\mathbf{jk}}\right\|\right) \leq \mathbb{E}\left(\sup_{|g|\leq 1} \left\|\left[g(\widetilde{M})\right]_{\mathbf{jk}}\right\|\right) \leq Ce^{-\eta|\mathbf{j}-\mathbf{k}|}$$

Thus,

$$\mathbb{E}\left(\sup_\alpha \mathcal{E}(\rho_\alpha)\right) \leq \frac{4\sqrt{2}\ C\ \log 2}{(e^\eta+1)^2}$$